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LETTER TO THE EDITOR

A labelling scheme for higher-dimensional simplex equations

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Abstract. We present a succinct way of obtaining all possible higher-dimensional generalizations of the quantum Yang–Baxter equation (QYBE). Using the scheme, we could generate the two popular three-simplex equations, namely Zamolodchikov's tetrahedron equation (ZTE) and Frenkel and Moore equation (FME).

The Quantum Yang–Baxter Equation (QYBE) is a nonlinear equation which appears in various forms in areas like integrable statistical models, topological field theories, the theory of braid groups, the theory of knots and links and conformal field theory.

Several higher-dimensional generalizations of QYBE have been proposed. By considering the scattering of straight strings in $2+1$ dimensions, Zamolodchikov proposed a higher-dimensional generalization of QYBE, commonly called the tetrahedron equation (ZTE) [1]:

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123} \tag{1}$$

where $R_{123} = R \otimes \mathbb{1}$ etc and $R \in \mathrm{End}(V \otimes V \otimes V)$ for some vector space V. As the QYBE is often called a two-simplex equation, this generalization of QYBE is a three-simplex equation. In general, there are d^{12} equations with d^6 variables, where d is the dimension of the vector space V. Despite its complexity, Zamolodchikov has ingeniously proposed a spectral-dependent solution which was subsequently confirmed by Baxter [1,2].

The tetrahedron equation is not the only possible higher-dimensional generalization. Frenkel and Moore [3] have proposed another higher-dimensional generalization (the FME), namely

$$R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123}\,. \tag{2}$$

They have also given an analytical solution for their three-simplex equation, namely

$$R = q^{1/2}q^{(h\otimes h\otimes h)/2}[1 + (q-q^{-1})(h\otimes e\otimes f + f\otimes e\otimes h - e\otimes hq^{-h}\otimes f)] \tag{3}$$

where e, f, h are generators of $sl(2)$ satisfying $[h,e] = 2e$, $[h,f] = 2f$, $[e,f] = h$. The three-simplex equation is not equivalent to Zamolodchikov's equation, as explained by Frenkel and Moore in their paper [3]. Furthermore, solutions of the three-simplex equation do not in general satisfy Zamolodchikov's tetrahedron equation (1). In fact, the

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expression (3) does not satisfy Zamolodchikov's tetrahedron equation unless the parameter q approaches unity, giving the identity matrix.

By considering the commutativity of the matrices $S_{i_1 i_2 \cdots i_d}^{j_1 j_2 \cdots j_d}$, Maillet and Nijhoff [4] were able to generalize the QYBE to higher-dimensional forms.

In this letter, we present a succinct way of writing these generalized equations. Consider a sequence $(1, 1, 0, \ldots, 0)$, in which all the entries are zeros except for two ones, and all its possible permutations. By interpreting each sequence as a binary number and arranging the resulting array in decreasing (or increasing) value, we can obtain a new array in which the 'transpose' generates the higher-dimensional forms of the QYBE.

As an example, consider the sequence $(1, 1, 0, 0)$. The resulting array obtained by arranging all possible permutations into an array of decreasing binary values is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} . \tag{4}$$

By writing rows as columns and columns as rows, we get a new array

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} . \tag{5}$$

Identifying the first row of the new matrix (5) as the transfer matrix R_{123} acting on the vector spaces $V \otimes V \otimes V \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ and so forth, we obtain $R_{123}R_{145}R_{246}R_{356}$, which is the left-hand side of equation (1).

Consider the 'mirror' image of (4): we get the array

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

and considering the transpose again, we get

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} . \tag{7}$$

The resulting array is the same as the array (5) but with the row order reversed. The first row is now identified as R_{356} and so forth, and the right-hand side of equation (1) is obtained.

We note that if we start with the starting sequence $(1, 1, 0)$, we get the QYBE. Furthermore, if we consider the sequence $(1, 1, 0, 0, 0)$, we get the Bazhanov–Stroganov equations [4], namely

$$R_{1,2,3,4}R_{1,5,6,7}R_{2,5,8,9}R_{3,6,8,10}R_{4,7,9,10} = R_{4,7,9,10}R_{3,6,8,10}R_{2,5,8,9}R_{1,5,6,7}R_{1,2,3,4} \, . \tag{8}$$

Higher-dimensional forms of the tetrahedron equation (1) can be obtained by considering sequences of the form $(1, 1, 0, \ldots, 0)$.

The Frenkel and Moore equation (2) and its higher-dimensional generalizations can be generated from starting sequences of the form $(1, 1, \ldots, 1, 0)$, in which all the entries except the last are ones. These sequences give trivial permutation arrays. In particular, the sequence $(1, 1, 1, 0)$ gives the Frenkel and Moore equation (2).

When the length of the starting sequence exceeds four, new simplex equations are obtained. For example, the sequence $(1, 1, 1, 0, 0)$ yields a new commutativity equation:

$$R_{1,2,3,4,5,6}R_{1,2,3,7,8,9}R_{1,4,5,7,8,10}R_{2,4,6,7,9,10}R_{3,5,6,8,9,10}$$
$$= R_{3,5,6,8,9,10}R_{2,4,6,7,9,10}R_{1,4,5,7,8,10}R_{1,2,3,7,8,9}R_{1,2,3,4,5,6}\,. \tag{9}$$

As we increase the length of the sequence, we find an increasing number of such higher-dimensional forms.

An alternative way of looking at the labelling scheme is to consider the non-commutative expansion

$$(x_1 + y_1)(x_2 + y_2)\cdots(x_n + y_n) = (y_1 + x_1)(y_2 + x_2)\cdots(y_n + x_n) \tag{10}$$

where we identify for instance, in the case $n = 4$, the term $x_1x_2y_3y_4$ as the sequence $(1, 1, 0, 0)$, setting $x_i = 1$ and $y_i = 0$.

As an example, if we consider the expansion

$$(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) = (x_3 + y_3)(x_2 + y_2)(x_1 + y_1)\,. \tag{11}$$

The left-hand side of the expansion gives

$$x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + y_1x_2x_3 + y_1x_2y_3 + y_1y_2x_3 + y_1y_2y_3. \tag{12}$$

Replacing x_i by 1 and y_i by 0 and grouping the terms with the same number of x's into arrays of the form (4), we get four different arrays, namely

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \tag{13}$$

corresponding to $x_1x_2x_3$,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{14}$$

corresponding to terms of the form $x_ix_jy_k$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

corresponding to terms of the form $x_iy_jy_k$, and

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \tag{16}$$

corresponding to terms of the form $y_iy_jy_k$.

The first array (13), the third array (15) and the fourth array (16) lead to trivial simplex equations, namely $R_{1234} = R_{1234}$, $R_1R_2R_3R_4 = R_4R_3R_2R_1$ and $\mathbb{1} = \mathbf{1}$. Only terms of the array (14) provide us with non-trivial simplex equations, namely $R_{12}R_{13}R_{23}$, the left-hand side of the QYBE. Expanding the right-hand side of (11) and grouping the terms $x_ix_jy_k$ will lead to $R_{23}R_{13}R_{12}$, the right-hand side of the QYBE.

The number of terms in each array corresponds to the coefficient of the binomial expansion $(x + y)^n$; in our previous example, corresponding to each array we have

$$\binom{3}{3} \quad \binom{3}{2} \quad \binom{3}{1} \quad \text{and} \quad \binom{3}{0}.$$

Thus it is not surprising that the number of vector spaces in which the higher-dimensional generalization of the QYBE acts are the coefficients of a binomial expansion. Furthermore, if we consider the expansion $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4)$, we see that for three-simplex case there are only two possible non-trivial generalizations, namely the ZTE and FME. Work is still in progress to understand more about the implications and significance of this labelling scheme; here we merely note that the Yang–Baxter equation is deeply rooted to the permutation group [5].

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